

# NONCONTINUOUS ADDITIVE ENTROPIES OF PARTITIONS

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We complete a description of **additive partition entropies**. This time we do not assume  $I$  to be continuous. In the process we solve a 2-cocycle functional equation for certain subsets of convex cones.

**1. Introduction.** We assume all definitions and results from [3]. In the current follow-up paper we drop the assumption of the continuity of the additive partition entropy. Our aim is to prove a theorem analogical to Theorem 1 of [3]. As a by-product of our efforts we obtain that all symmetric solutions to a 2-cocycle functional equation for certain subsets of convex cones are coboundaries (Lemma 2, see also Remark 1).

**2. The results.** To begin with we need to extend Example 1 of [3].

By  $\text{End}(\mathbb{R})$  we shall mean the  $\mathbb{Q}$ -algebra of endomorphisms of the linear space  $\mathbb{R}$  of reals over the field  $\mathbb{Q}$  of rationals.

EXAMPLE 1. Consider a finitely-additive set function  $\mathbf{m}: \mathcal{X} \rightarrow \text{End}(\mathbb{R})$  vanishing on sets of  $P$ -measure 0. Then the mapping  $L_{\mathbf{m}}: \mathfrak{A} \rightarrow \mathbb{R}$  defined by

$$L_{\mathbf{m}}(A) := \sum_{1 \leq i \leq n} \mathbf{m}(A_i) \left( \log \frac{1}{P(A_i)} \right),$$

where  $A = \langle A_1, \dots, A_n \rangle$ , is an additive partition entropy.

THEOREM 1. Let  $I$  be an additive partition entropy. There exist an additive entropy  $H$ , and a finitely-additive set function  $\mathbf{m}: \mathcal{X} \rightarrow \text{End}(\mathbb{R})$  vanishing on sets of  $P$ -measure 0 such that

$$I = H_P + L_{\mathbf{m}}.$$

PROOF. According to Proposition 4 in [3], given any sets  $V, W$  of the same measure  $P(V) = P(W)$  one can define a mapping  $\Delta(V, W) \in \text{End}(\mathbb{R})$  as  $\log \lambda \mapsto \Delta(V, W, \lambda)$ .  $\Delta$  satisfies the following conditions:

1. For any sets  $U, V$  and  $W$  of the same measure

$$\Delta(U, W) = \Delta(U, V) + \Delta(V, W).$$

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This paper is partially supported by Grant nr N N201 605840.

AMS 2000 subject classifications: Primary 94A17; secondary 60A10

Keywords and phrases: additive entropy, inset entropy, partition entropy, axioms of entropy, independent  $\sigma$ -algebras

2. For any sequences of disjoint sets  $A, B, \dots, C$  and  $A', B', \dots, C'$  which satisfy the equalities  $P(A) = P(A'), \dots, P(C) = P(C')$  we have

$$\Delta(A + B + \dots + C, A' + B' + \dots + C') = \Delta(A, A') + \Delta(B, B') + \dots + \Delta(C, C').$$

3. Moreover by Remark 5 in [3], if  $P(V \Delta V') = P(W \Delta W') = 0$ , then

$$\Delta(V, W) = \Delta(V', W').$$

By analogy with Theorem 1 in [3] what we need is the following Lemma, which in some way can be regarded as a strengthening of Lemma 6 there

LEMMA 1. *For any  $\mathbb{Q}$ -linear space  $L$  and any  $\Delta(V, W) \in L$ , defined for sets  $V, W$  of the same measure  $P$  and satisfying conditions 1–3 above there is a finitely-additive set function  $m: \mathcal{X} \rightarrow L$  such that  $m(V) = 0$  whenever  $P(V) = 0$  and also such that for any sets  $V$  and  $W$  satisfying  $P(V) = P(W)$  we have*

$$\Delta(V, W) = m(W) - m(V).$$

PROOF. To begin with, notice that there is a mapping  $m_1: \mathcal{X} \rightarrow L$  such that we have  $\Delta(V, W) = m_1(W) - m_1(V)$  and  $m_1(V) = 0$  when  $P(V) = 0$ . Indeed, for any  $\theta \in [0, 1]$  choose an arbitrary  $V_\theta$  such that  $P(V_\theta) = \theta$  and set  $m_1(V_\theta) = 0$ . Next, for any  $W$  of measure  $P(W) = \theta$ , set  $m_1(W) = \Delta(V_\theta, W)$ . Conditions 1 and 3 above give  $\Delta(V, W) = m_1(W) - m_1(V)$  for any  $V, W \in \mathcal{X}$ ,  $P(V) = P(W)$ .

By property 2 above the number

$$m_1(A + B + \dots + C) - m_1(A) - m_1(B) \dots - m_1(C),$$

defined for disjoint  $A, B, \dots, C \in \mathcal{X}$ , depends solely on measures

$$P(A), P(B), \dots, P(C),$$

i.e. there is a function

$$f: \{(a, b, \dots, c) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \dots \times \mathbb{R}_+ : a + b + \dots + c \leq 1\} \rightarrow L$$

such that

$$(1) \quad m_1(A + B + \dots + C) - m_1(A) - m_1(B) \dots - m_1(C) = f(P(A), P(B), \dots, P(C)).$$

We see that  $f$  has the following properties

$$(2) \quad \begin{aligned} f(0, 0, \dots, 0) &= 0, \\ f(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}) &= f(a_1, a_2, \dots, a_n), \end{aligned}$$

for any permutation  $\sigma$ , and also, as a direct consequence of (1),

$$(3) \quad \begin{aligned} f(a_1, \dots, a_k, b_1, \dots, b_l, \dots, c_1, \dots, c_m) &= \\ f(a_1 + \dots + a_k, b_1 + \dots + b_l, \dots, c_1 + \dots + c_m) &= \\ + f(a_1, \dots, a_k) + f(b_1, \dots, b_l) + \dots + f(c_1, \dots, c_m). \end{aligned}$$

Suppose for now we can find a mapping  $h: [0, 1] \rightarrow L$  satisfying

$$f(a, b, \dots, c) = h(a + b + \dots + c) - h(a) - h(b) \dots - h(c).$$

Then  $h(0) = -f(0, 0) = 0$ . Moreover, setting  $m(V) := m_1(V) - h(P(V))$  we obtain a new function  $m: \mathcal{X} \mapsto L$ . It follows straight from (1) that  $m$  is finitely-additive. What is more, if  $P(V) = P(W)$ , then

$$m(W) - m(V) = m_1(W) - m_1(V) = \Delta(V, W).$$

The possibility of finding  $h$  given (2) and (3) is a ‘homological’ fact which, being interesting in itself, we have separated as Lemma 2 below.  $\square$

LEMMA 2. *Consider any  $\mathbb{Q}$ -linear spaces  $K$  and  $L$ , a convex cone  $P \subset K$  such that  $P \cap -P = \{0\}$  and a set  $M \subset P$  such that  $P = \mathbb{Z}_+ M$  and let  $a, b \in P$ ,  $a + b \in M$  imply  $a, b \in M$ .*

*For any function*

$$f: \bigcup_{n \geq 1} \{(a_1, a_2, \dots, a_n) \in P \times P \times \dots \times P : a_1 + a_2 + \dots + a_n \in M\} \rightarrow L,$$

*the values of which do not depend on the permutation of arguments:*

$$(4) \quad f(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(n)}) = f(a_1, a_2, \dots, a_n),$$

*and which also satisfies*

$$(5) \quad \begin{aligned} f(a_1, \dots, a_k, b_1, \dots, b_l, \dots, c_1, \dots, c_m) = \\ f(a_1 + \dots + a_k, b_1 + \dots + b_l, \dots, c_1 + \dots + c_m) \\ + f(a_1, \dots, a_k) + f(b_1, \dots, b_l) + \dots + f(c_1, \dots, c_m), \end{aligned}$$

*there is a function  $h: M \rightarrow L$  such that for  $n \geq 1$*

$$(6) \quad f(a_1, a_2, \dots, a_n) = h(a_1 + a_2 + \dots + a_n) - h(a_1) - h(a_2) \dots - h(a_n).$$

PROOF. For a single argument, (5) gives  $f(a) = 0$ . From a repeated application of (5) we also get

$$f(a_1, \dots, a_n) = f(a_1 + \dots + a_{n-1}, a_n) + f(a_1 + \dots + a_{n-2}, a_{n-1}) + \dots + f(a_1, a_2).$$

Thus (6) will follow once we show that  $h$  satisfies

$$(7) \quad f(a, b) = h(a + b) - h(a) - h(b).$$

Observe that for any  $s \in M$ ,  $\alpha \in \mathbb{Q}$ ,  $\alpha \leq 1$  we have  $\alpha s, (1 - \alpha)s \in P$ ,  $\alpha s + (1 - \alpha)s = s \in M$ , thus  $\alpha s \in M$ .

In order to shorten notation we shall write  $a : k$  for  $\overbrace{a, \dots, a}^k$ . From (5) and in particular from  $f(0) = 0$  we find that there is a constant  $z \in L$  such that

$f(0 : n) = (n-1)z$ ,  $n \geq 1$ . By (7) any solution  $h$  must have  $f(a, 0) + h(0) = 0$  and  $h(0) = -z$ .<sup>1</sup>

For any  $s \in M \setminus \{0\}$  and for the ray  $S = \mathbb{Q}_+s$ , there exists a solution  $h_S : S \cap M \rightarrow L$  to equation (7) restricted to  $a, b, a+b \in S \cap M$ , with arbitrary value  $h_S(s) \in L$ . Indeed, begin by writing  $h_S(0) = -z$ , choose arbitrarily  $h_S(s) \in L$  and then extend

$$(8) \quad h_S\left(\frac{p}{q}s\right) := \frac{p}{q} \left[ h_S(s) - f\left(\frac{s}{q} : q\right) \right] + f\left(\frac{s}{q} : p\right),$$

for any positive integers  $p, q$  such that  $\frac{p}{q}s \in M$ . The function  $h_S$  is well-defined since for any  $n \geq 2$ , equality (5) gives us

$$\begin{aligned} f\left(\frac{s}{qn} : pn\right) - \frac{p}{q} f\left(\frac{s}{qn} : qn\right) &= \\ &= \left[ f\left(\frac{s}{q} : p\right) + pf\left(\frac{s}{qn} : n\right) \right] - \frac{p}{q} \left[ f\left(\frac{s}{q} : q\right) + qf\left(\frac{s}{qn} : n\right) \right] \\ &= f\left(\frac{s}{q} : p\right) - \frac{p}{q} f\left(\frac{s}{q} : q\right). \end{aligned}$$

Moreover,  $h_S$  satisfies (7). In fact, given positive  $k, l$ , from

$$f\left(\frac{s}{q} : k+l\right) = f\left(\frac{k}{q}s, \frac{l}{q}s\right) + f\left(\frac{s}{q} : k\right) + f\left(\frac{s}{q} : l\right),$$

we have by (8)

$$h_S\left(\frac{k+l}{q}s\right) = f\left(\frac{k}{q}s, \frac{l}{q}s\right) + h_S\left(\frac{k}{q}s\right) + h_S\left(\frac{l}{q}s\right),$$

Which is also true if any of  $k, l$  is 0. Since equality (7) implies formula (8) there is only one such solution  $h_S$  having a fixed value at a given nonzero point  $s \in S \cap M$ .

Suppose that for some convex cone  $C \subset P$  we are given a solution  $h_1 : C \cap M \rightarrow L$  to equation (7) restricted to  $a, b, a+b \in C \cap M$ . As we shall presently show, for any  $s \in M \setminus C$  there is an extension  $h : (C \vee \mathbb{Q}_+s) \cap M \rightarrow L$  which continues to satisfy equation (7) with  $a, b, a+b \in (C \vee \mathbb{Q}_+s) \cap M$ .

Suppose first that  $s \notin \text{Span}(C)$ . Any element of  $(C \vee \mathbb{Q}_+s) \cap M$  can be then uniquely represented as  $a+b \in M$ , with  $a \in C$ ,  $b \in \mathbb{Q}_+s$ . Define for such an element

$$(9) \quad h(a+b) := h_1(a) + h_S(b) + f(a, b),$$

where  $h_S$  is any solution to (7) on  $\mathbb{Q}_+s \cap M$ . In particular,  $h(a) = h_1(a)$ ,  $h(b) = h_S(b)$ . What is more, for any  $a, a' \in C$  and  $b, b' \in \mathbb{Q}_+s$ , such that  $a+a'+b+b' \in M$  we obtain

$$\begin{aligned} h(a+b+a'+b') &= \\ &= h(a+a') + h(b+b') + f(a+a', b+b') && \text{(by (9))} \\ &= h(a) + h(a') + h(b) + h(b') + f(a, a', b, b') && \text{(by (7), (5))} \\ &= h(a+b) + h(a'+b') + f(a+b, a'+b'). && \text{(by (9), (5))} \end{aligned}$$

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<sup>1</sup>

By replacing  $f(a_1, a_2, \dots, a_n)$  with  $f(a_1, a_2, \dots, a_n) - (n-1)z$  and  $h(a)$  with  $h(a) + z$  we could assume that  $f(0, 0, \dots, 0) = 0$ . Then we would have  $f(a, 0) = 0$  and  $h(0) = 0$ .

Suppose now that  $s \in \text{Span}(\mathbb{C})$ . In this case there are  $r \in \mathbb{C}$ ,  $\alpha_0 \in \mathbb{Q}_+$  with  $r + s \in \mathbb{C}$  and  $\alpha_0(r + s) \in M$ . Consider a solution  $h_S$  to equation (7) on  $\mathbb{Q}_+s \cap M$  with  $h_S(\alpha_0s)$  chosen in such a way so as to satisfy

$$(10) \quad h_1(\alpha_0(r + s)) = h_1(\alpha_0r) + h_S(\alpha_0s) + f(\alpha_0r, \alpha_0s).$$

Now, observe that grouping  $\alpha r, \beta r, \alpha s, \beta s$  in two different ways gives by (5)

$$(11) \quad \begin{aligned} f(\alpha r, \beta r, \alpha s, \beta s) &= \\ &= f(\alpha(r + s), \beta(r + s)) + f(\alpha r, \alpha s) + f(\beta r, \beta s) \\ &= f((\alpha + \beta)r, (\alpha + \beta)s) + f(\alpha s, \beta s) + f(\alpha r, \beta r) \end{aligned}$$

whenever  $(\alpha + \beta)(r + s) \in M$ . It follows that the mapping  $H_S$  defined for such  $\alpha s$  that  $\alpha(r + s) \in M$  by

$$H_S(\alpha s) := h_1(\alpha(r + s)) - h_1(\alpha r) - f(\alpha r, \alpha s)$$

satisfies

$$\begin{aligned} H_S(\alpha s + \beta s) - H_S(\alpha s) - H_S(\beta s) &= \\ &= f(\alpha(r + s), \beta(r + s)) - f(\alpha r, \beta r) && \text{(by (7) for } h_1) \\ &\quad - f((\alpha + \beta)r, (\alpha + \beta)s) + f(\alpha r, \alpha s) + f(\beta r, \beta s) \\ &= f(\alpha s, \beta s) && \text{(by (11))} \end{aligned}$$

i.e. condition (7) on the set  $M' = \{\alpha s : \alpha(r + s) \in M, \alpha \in \mathbb{Q}_+\}$ . Since  $M'$  plays the role of  $M$  for the ray  $\mathbb{Q}_+s$  and  $H_S(\alpha_0s) = h_S(\alpha_0s)$ ,  $H_S$  is a restriction of  $h_S$  to  $M'$ . This shows that

$$(12) \quad h_1(\alpha(r + s)) = h_1(\alpha r) + h_S(\alpha s) + f(\alpha r, \alpha s),$$

for any  $\alpha \in \mathbb{Q}_+$ ,  $\alpha(r + s) \in M$ . Define  $h$  by

$$h(a + b) := h_1(a) + h_S(b) + f(a, b),$$

for  $a \in \mathbb{C}$ ,  $b \in \mathbb{Q}_+s$ ,  $a + b \in M$ . It remains to check that  $h$  is well-defined, and then to proceed like in the previous case with  $s \in \text{Span}(\mathbb{C})$ . To this end, let  $a' - a = b - b' = \alpha s$ , where  $a' \in \mathbb{C}$ ,  $b' \in \mathbb{Q}_+s$ . We can assume that  $\alpha \geq 0$ . Then for  $\alpha r := a$  equality (12) takes the following form:

$$h_1(a') = h_1(a) + h_S(b - b') + f(a, b - b').$$

Additionally  $f(a, b - b') + f(a', b') = f(a, b - b', b') = f(b - b', b') + f(a, b)$ . Therefore

$$\begin{aligned} h_1(a') + h_S(b') + f(a', b') &= \\ &= h_1(a) + h_S(b') + h_S(b - b') + f(a, b - b') + f(a', b') \\ &= h_1(a) + h_S(b') + h_S(b - b') + f(b - b', b') + f(a, b) \\ &= h_1(a) + h_S(b) + f(a, b). && \text{(by (7) for } h_S) \end{aligned}$$

The existence of  $h$  satisfying (7) for  $a, b, a + b \in (C \vee Q_+ s) \cap M$  is thus proved.

Consider the family  $\mathcal{H}$  of the solutions  $h$  to equation (7), defined on the intersections of shape  $C \cap M$ , where  $C \subset P$  is a convex cone.  $\mathcal{H}$  is trivially nonempty and inductive, (with respect to the partial order defined by  $h_1 \preceq h_2 \iff h_1 \subset h_2$ ). By Kuratowski-Zorn Lemma the family  $\mathcal{H}$  has a maximal element  $h_C: C \cap M \rightarrow L$ . We shall show by contraposition that  $M \subset C$ . Indeed, assume that we can find  $s \in M$ ,  $s \notin C$ . It allows us, according to previous considerations, to define a mapping  $h \in \mathcal{H}$  on the set  $(C \vee Q_+ s) \cap M$  which extends  $h_C$ .  $\square$

This concludes the proof of Theorem 1.  $\square$

It should be possible to derive a version of Lemma 2 with considerably weaker assumptions on the shape of  $M$ . It seems, however, that it would make the proof longer still.

The remark below makes it easy to spot a relation between the function  $f$  of the last Lemma and other similar notions that appear elsewhere. In fact, it shows that the Lemma is a variant of a result by Erdős [2], that a symmetric 2-cocycle is a coboundary, see [1].

REMARK 1. *In the assumptions of Lemma 2 it suffices to define  $f$  for two arguments and replace (4), (5) with the conditions*

$$(13) \quad \begin{aligned} f(a, b) &= f(b, a), \\ f(a, b + c) + f(b, c) &= f(a + b, c) + f(a, b). \end{aligned}$$

In fact, we extend the definition of  $f$  by setting:

$$\begin{aligned} f(a) &:= 0 \\ f(a, b, \dots, c) &:= f(a + b, \dots, c) + f(a, b), \end{aligned}$$

for any  $a, b, \dots, c \in P$  such that  $a + b + \dots + c \in M$ .

We have to show (4) and (5). In fact, the former is obvious when there are at most three arguments. For  $n > 3$  arguments we use induction. By the definition of  $f$  it follows, that we can permute the last  $n - 2$  arguments without changing the value of  $f$ . Since, by definition of  $f$  for the sequences  $(a, b, c, \dots, d)$ ,  $(a + b, c, \dots, d)$  and  $(a, b, c)$  we have

$$f(a, b, c, \dots, d) = f(a + b + c, \dots, d) + f(a, b, c),$$

we can also permute the first 3 arguments. This shows (4). Coming over to (5) it suffices to show it in case when one of the numbers  $k, l, \dots, m$  is greater than 1. We can assume that  $k \geq 2$ . Then the sought-after equality follows from the same equality for the shorter sequence

$$(a_1 + a_2, \dots, a_k, b_1, \dots, b_l, \dots, c_1, \dots, c_m)$$

and the definitions of  $f$  for both sequences  $(a_1, \dots, a_k)$  and

$$(a_1, \dots, a_k, b_1, \dots, b_l, \dots, c_1, \dots, c_m).$$

A reasoning similar to that of Remark 1 in [3] gets us the following

REMARK 2. *The additive partition entropy of form  $L_m$ , for a finitely-additive set function  $m: \mathcal{X} \rightarrow \text{End}(\mathbb{R})$  vanishing on sets of  $P$ -measure 0, depends solely on measures of atoms exactly when there is an  $\alpha \in \text{End}(\mathbb{R})$  such that  $m(A) = \alpha(P(A))$  for any  $A \in \mathcal{X}$ .*

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